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The Uniform Motion of a Sphere through a Viscous Liquid.

BY R. W. BURGESS.

1. The problem of finding at any point in a viscous liquid the velocity due to the uniform rectilinear motion of a sphere was first attacked by Stokes in 1851 and an approximate solution obtained. From this was derived the well-known Stokes' law, that a force $6\pi\mu W a$ is required to maintain a uniform velocity W of a sphere of radius a through a liquid whose coefficient of viscosity is μ . This solution was proposed merely as a limiting form to which the distribution of velocities may be supposed to tend as W approaches the limit zero, but has nevertheless proved itself well in accord with experimental facts, at least as a close first approximation. The serious mathematical incompleteness of the solution has been recognized, however, and several attempts have been made to improve on it. Whitehead, in 1888, attempted to modify this solution by taking into account terms depending on squares of velocities, but found it impossible by his method to satisfy the boundary conditions at infinity. In 1893, Rayleigh showed that Stokes' solution would be accurate if certain additional forces were introduced, the magnitude of the required forces furnishing some indication of the error of the solution. In 1911, Oseen found a solution which is satisfactory at infinity; it also gives a good approximation near the sphere if the velocity or the radius of the sphere is small. Shortly afterwards Lamb, using a different method, presented Oseen's results in a simpler form, but still subject to the same limitations. In 1913, F. Noether attempted to take into consideration the terms neglected by Stokes, but merely derived Whitehead's correction over again. A few months later Oseen took up the same problem and tried to apply the principle of his previous solution, but fell into the same error that he had carefully pointed out in Whitehead's work. In the following paper Oseen's results are obtained by a simpler method than that of either Oseen or Lamb. The method is generalized, so as to give a solution that satisfies the boundary conditions at the sphere to any desired degree of approximation. A correction is determined which takes account of all the terms quadratic in velocities, and differs from the corrections of Whitehead, Noether and Oseen in that it satisfies the boundary conditions at infinity.

2. Using cylindrical coordinates (r, z) , let u, w denote the components of the velocity, p the pressure, d the density, $\nu \left(= \frac{\mu}{d} \right)$ the kinematic coefficient of viscosity, and V the potential of the impressed forces. The differential equations for the motion of a viscous liquid, in the case when the Z -axis is an axis of symmetry, are*

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} &= \frac{\partial Q}{\partial r} + \nu \left(\nabla^2 u - \frac{u}{r^2} \right), \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} &= \frac{\partial Q}{\partial z} + \nu \nabla^2 w, \end{aligned} \right\} \quad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$Q \equiv -V - p/d.$$

Eliminating Q from these equations, we obtain

$$\begin{aligned} \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) - \frac{\partial}{\partial r} \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) \\ = \nu \left[\frac{\partial}{\partial z} \left(\nabla^2 u - \frac{u}{r^2} \right) - \frac{\partial}{\partial r} \nabla^2 w \right]. \end{aligned} \quad (2)$$

This equation is simplified by making use of the equation of continuity†

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0.$$

If we put

$$u = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad (3)$$

this relation is satisfied; hence the velocities may be found by differentiation from this function ψ , known as Stokes' current function, and the problem is reduced to the determination of ψ . The curves of intersection of planes through the z -axis with the surface $\psi = \text{a constant}$, will be the lines of flow or stream lines; that is, the lines indicating the direction of the resultant velocity at any point. This system of curves provides a convenient method of representing graphically the motion given by any function ψ .

* Bassett, "Hydrodynamics," Vol. II, p. 244.

† Bassett, "Hydrodynamics," Vol. I, pp. 8, 12.

Substituting in (2) the values of u and w given in (3), we have*

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z} - \frac{2u}{r} - \nu D\right) D\psi = 0, \quad (4)$$

where

$$D \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The preceding equation applies to any case in which the motion of the liquid is symmetrical with respect to the z -axis. We desire to discuss in this paper the motion due to a sphere which has been moving for an infinite time with uniform velocity W along the z -axis, and whose center is at $(0, Wt)$ at time t . Since the conditions at a point depend only on its position relative to the sphere, ψ is evidently a function of r and $z - Wt$ only. If $z' = z - Wt$, we have

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial z'} \frac{\partial z'}{\partial z} = \frac{\partial \psi}{\partial z'}, \quad \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial z'} \frac{\partial z'}{\partial t} = -W \frac{\partial \psi}{\partial z'}.$$

Equation (4) then becomes

$$\left(u \frac{\partial}{\partial r} + (w - W) \frac{\partial}{\partial z'} - \frac{2u}{r} - \nu D\right) D\psi = 0, \quad (5)$$

a form on which we shall base our further discussion.

3. The problem we shall consider is the determination of the function ψ of r and z' which satisfies (5) and is such that the velocity of the liquid at the surface of the sphere is the same as the velocity of the sphere, and that the velocity of the liquid at an infinite distance from the sphere vanishes. This problem was first attacked by Stokes† in 1851 and solved, approximately, for the case in which the liquid is very viscous; that is, ν large, and the velocities sufficiently small so that the terms involving the products of velocities and accelerations are comparatively unimportant. On these assumptions the approximate form of (5) is

$$\nu D \cdot D\psi = 0, \quad (6)$$

for which we may easily obtain Stokes' solution

$$\psi = \frac{1}{4} W a^2 \left(\frac{3\rho}{a} - \frac{a}{\rho} \right) \sin^2 \theta,$$

where a is the radius of the sphere, $\rho = \sqrt{r^2 + z'^2}$ and

$$\tan \theta = \frac{r}{z'}. \quad (7)$$

* Cf. Bassett, "Hydrodynamics," Vol. II, p. 259.

† *Cambridge Transactions*, Vol. IX (1851); *Sci. Papers*, Vol. III, p. 55.

From this is derived Stokes' well-known formula $F=6\pi\mu Wa$ for the force required to maintain the motion. This formula has proved so accurate for small velocities that the approximations by which it was obtained have not always been fully recognized. Stokes himself, however, proposed the solution (7) only as a limiting form to which the velocities may be supposed to tend as W approaches zero.

In 1888, Whitehead* attempted to modify this solution to take account of the neglected quadratic terms. The attempt was successful in finding correction terms satisfactory as far as the differential equation (5) and the boundary conditions at the sphere were concerned, but these terms failed, as Whitehead himself noticed, to satisfy the condition that the velocity at an infinite distance vanishes. The correction terms for the velocities along and perpendicular to the radius vector were found to be

$$\left. \begin{aligned} R_1 &= -\frac{3}{16} \frac{W^2 a}{\nu} \left(\frac{a}{\rho} - 1 \right)^2 \left(\frac{a^2}{\rho^2} + \frac{a}{\rho} + 2 \right) P_2(\cos \theta), \\ \Theta_1 &= -\frac{3}{16} \frac{W^2 a}{\nu} \left(\frac{a}{\rho} - 1 \right) \left(\frac{a^3}{\rho^3} + \frac{a^2}{2\rho^2} + \frac{a}{2\rho} + 2 \right) \sin \theta \cos \theta. \end{aligned} \right\} \quad (8)$$

It is obvious that when ρ is infinite these terms do not vanish as required.

In 1893, Rayleigh† showed that Stokes' solution would be accurate if certain additional forces were introduced; the ratio of the forces thus required to the viscous forces actually considered gives a means of estimating the accuracy of the approximation. This ratio can easily be shown to be $\frac{W\rho}{\nu}$, which increases indefinitely with ρ ; Stokes' solution is therefore not valid at infinity.

4. In 1911, Oseen‡ introduced a new approximate solution, valid at infinity. Consider the term $(w-W) \frac{\partial}{\partial z}$ in equation (5). Near the surface of the sphere it is true that $w-W$ is very small, and the neglect of this term is justified, but at any considerable distance it is not; in fact, $w-W$ is very nearly $-W$, not zero. Oseen shows that the ratio of the neglected terms to the terms giving the viscous forces is $\frac{Wz}{\nu}$, which becomes infinite with z , no matter how small we take W . On account of these considerations at a large distance from the sphere, we take as an approximate form of (5)

$$\left(-W \frac{\partial}{\partial z'} - \nu D \right) D\psi = 0. \quad (9)$$

* *Quarterly Journal of Mathematics*, Vol. XXIII (1888), pp. 143-152.

† *Philosophical Magazine*, Ser. 4, Vol. XXXVI (1893), p. 354.

‡ *Arkiv för Matematik, Astronomi, och Fysik*, Bd. 6 (1911), No. 29.

Oseen's method is not founded on a consideration of equation (5) or (9), nor does he use the current function; but the above remarks embody his argument. He obtains a solution which, expressed as a current function, satisfies (9) and the boundary conditions at infinity, but satisfies the boundary conditions at the surface of the sphere only to the first power of the quantity $\frac{Wa}{\nu}$, which is assumed to be small.

5. The essential points of this work of Oseen were restated by Lamb* shortly afterwards, with the use of a less intricate method of analysis. Lamb points out especially the significance of the result, showing that the equation (9), or rather the similar equations used by Oseen, represents more accurately than does Stokes' equation (6) the nature of the motion at a large distance from the sphere. Lamb shows, moreover, that Oseen's solution more nearly represents the wake which follows a moving object, in that the velocity preceding the sphere is markedly smaller than that following.

Two years later F. Noether,† referring in his work to that of Oseen, determined as a correction to Stokes' current function (7)

$$\psi_1 = \frac{3}{32} \frac{W^2 a}{\nu} \sin^2 \theta \cos \theta \left(\frac{\rho - a}{\rho} \right)^2 (2\rho^2 + a\rho + a^2). \quad (10)$$

Corresponding to this, the corrections for the velocities are

$$R_1 = -\frac{3}{16} \frac{W^2 a}{\nu} P_2(\cos \theta) \left(\frac{\rho - a}{\rho} \right)^2 \left(2 + \frac{a}{\rho} + \frac{a^2}{\rho^2} \right),$$

$$\Theta_1 = \frac{3}{16} \frac{W^2 a}{\nu} \sin \theta \cos \theta \left(1 - \frac{a}{\rho} \right) \left(2 + \frac{a}{2\rho} + \frac{a^2}{2\rho^2} + \frac{a^3}{\rho^3} \right),$$

which are exactly (8), as found by Whitehead.

In the same year Oseen‡ again published an article on the subject criticizing certain portions of Noether's work, and giving as the value of Stokes' current function

$$\psi_2 = \frac{W}{4} \sin^2 \theta \left(1 - \frac{a}{\rho} \right)^2 \left[\left(1 + \frac{3}{8} \left| \frac{Wa}{\nu} \right| \right) \rho (2\rho + a) + \frac{3}{8} \frac{Wa}{\nu} \cos \theta (2\rho^2 + a\rho + a^2) \right] - \frac{W}{2} \rho^2 \sin^2 \theta. \quad (11)$$

But this function yields values of R and Θ which fail to vanish at infinity,

* *Philosophical Magazine*, Ser. 6, Vol. XXI (1911), pp. 112-121.

† *Zeitschrift für Mathematik und Physik*, Bd. 62.

‡ *Arkiv för Matematik, Astronomi, och Fysik*, Bd. 9, No. 16.

exactly as Whitehead's resolution did, and therefore, as Oseen pointed out in his earlier paper, is not admissible.

6. We shall now consider in detail equation (9),

$$\left(-W \frac{\partial}{\partial z'} - \nu D\right) D\psi = 0. \quad (9)$$

As these operations are commutative, a solution will be given by

$$\psi = \psi' + \psi'', \quad (12)$$

where

$$D\psi' = 0 \quad (13)$$

and

$$\left(\nu D + W \frac{\partial}{\partial z'}\right) \psi'' = 0. \quad (14)$$

We shall obtain the solution of these separately. Equation (13) has been treated very fully by Sampson,* but for convenience of reference we shall here give in detail the solution so far as is necessary for our purposes. Transforming to polar coordinates

$$\rho = \sqrt{r^2 + z'^2}, \quad \theta = \cos^{-1} \frac{z'}{\sqrt{r^2 + z'^2}} = \sin^{-1} \frac{r}{\sqrt{r^2 + z'^2}},$$

our operation D becomes

$$\frac{\partial^2}{\partial \rho^2} + \frac{\sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right).$$

We desire, then, the solution of the equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{\sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) = 0. \quad (13)$$

A particular solution is

$$\psi_n = c_n \frac{\sin \theta}{\rho^n} \frac{\partial P_n}{\partial \theta}, \quad (13A)$$

where P_n is the zonal harmonic of degree n , and therefore satisfies the differential equation

$$n(n+1)P_n \sin \theta + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P_n}{\partial \theta} \right) = 0. \quad (15)$$

For substituting in (13) the given value of ψ_n , we have for the left-hand side

$$\frac{n(n+1)c_n}{\rho^{n+2}} \sin \theta \frac{\partial P_n}{\partial \theta} + \frac{c_n \sin \theta}{\rho^{n+2}} \frac{\partial}{\partial \theta} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P_n}{\partial \theta} \right) \right\};$$

* *Philosophical Transactions*, Vol. CLXXXII (1891), p. 449.

that is, using (15),

$$\frac{c_n \sin \theta}{\rho^{n+2}} \left[n(n+1) \frac{\partial P_n}{\partial \theta} - \frac{\partial}{\partial \theta} \{ n(n+1) P_n \} \right],$$

which is identically zero. It is well known that

$$P_n = A_n \left\{ \cos^n \theta - \frac{n(n-1)}{2(2n-1)} \cos^{2n-2} \theta + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} \cos^{n-4} \theta - \dots \right\}. \quad (16)$$

Hence, by (14),

$$\begin{aligned} \psi_n = \frac{c_n \sin^2 \theta}{\rho^n} & \left[\cos^{n-1} \theta - \frac{(n-1)(n-2)}{2(2n-1)} \cos^{n-3} \theta + \dots \right. \\ & \left. + (-1)^r \frac{(n-1)(n-2) \dots (n-2r)}{2^r r(2n-1) \dots (2n-2r+1)} \cos^{n-2r+1} \theta + \dots \right] \end{aligned} \quad (13B)$$

for all positive integral values of n . For $n=0$, we have ψ a function of θ only; then

$$\frac{1}{\sin \theta} \frac{\partial \psi_0}{\partial \theta} = c_0, \quad \psi_0 = c_0 \cos \theta.$$

Placing n successively 1, 2, 3, ... in (13B), we have

$$\left. \begin{aligned} \psi_1 &= c_1 \frac{\sin^2 \theta}{\rho}, \\ \psi_2 &= c_2 \frac{\sin^2 \theta \cos \theta}{\rho^2}, \\ \psi_3 &= c_3 \frac{\sin^2 \theta (5 \cos^2 \theta - 1)}{\rho^3}, \\ \psi_4 &= c_4 \frac{\sin^2 \theta \cos \theta (7 \cos^2 \theta - 3)}{\rho^4}. \end{aligned} \right\} \quad (17)$$

7. We desire also the solution of equation (14), which is, if $2k = \frac{W}{\nu}$,

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z'^2} + 2k \frac{\partial}{\partial z'} \right) \psi'' = 0. \quad (14A)$$

Letting $\psi'' = e^{-kz'} \phi(r, z')$, we have at once

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z'^2} - k^2 \right) \phi = 0,$$

or, in polar coordinates,

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{\sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \phi}{\partial \theta} \right) - k^2 \phi = 0. \quad (18)$$

Assume $\phi = R\Theta$, where R and Θ are respectively functions of ρ and θ only.

$$\Theta \frac{\partial^2 R}{\partial \rho^2} + R \frac{\sin \theta}{\rho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Theta}{\partial \theta} \right) - k^2 R \Theta = 0,$$

$$\frac{\rho^2}{R} \frac{\partial^2 R}{\partial \rho^2} - k^2 \rho^2 = - \frac{\sin \theta}{\Theta} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Theta}{\partial \theta} \right) = n(n+1), \text{ say.}$$

We have, then, to determine R and Θ , the two equations:

$$\frac{\partial^2 R}{\partial \rho^2} - k^2 R - \frac{n(n+1)}{\rho^2} R = 0, \quad (19)$$

$$n(n+1)\Theta + \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Theta}{\partial \theta} \right) = 0. \quad (20)$$

In this form, for n any positive integer, R is a solution in finite form of Riccati's equation; that is,

$$R_n = \rho^{n+1} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \right)^{n+1} (Ae^{k\rho} + Be^{-k\rho}), \quad (21)$$

and Θ is $\sin \theta \frac{\partial P_n}{\partial \theta}$. We have, therefore,

$$\psi'' = \sum_{n=0}^{n=\infty} e^{-k\rho \cos \theta} \sin \theta \frac{\partial P_n}{\partial \theta} R_n. \quad (22)$$

We have, then, as a general function satisfying the differential equation (9)

$$\begin{aligned} \psi = e^{-k\rho(1+\cos \theta)} & \left[C + D \cos \theta + b_1 \sin^2 \theta \left(k + \frac{1}{\rho} \right) \right. \\ & \left. + b_2 \sin^2 \theta \cos \theta \left(k^2 + \frac{3k}{\rho} + \frac{3}{\rho^2} \right) + \dots \right] \\ & + L \cos \theta + \frac{M \sin^2 \theta}{\rho} + \frac{N \sin^2 \theta \cos \theta}{\rho^2} + \frac{P \sin^2 \theta}{\rho^3} (5 \cos^2 \theta - 1) + \dots \end{aligned} \quad (23)$$

8. To adapt this solution to the problem of the sphere moving with uniform velocity W , we must have, if

$$\left. \begin{aligned} R &= \frac{1}{\rho^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \text{ is the radial velocity} \\ \text{and } \Theta &= - \frac{1}{\rho \sin \theta} \frac{\partial \psi}{\partial \rho} \text{ is the velocity perpendicular to this,} \end{aligned} \right\} \quad (24)$$

$$\left. \begin{aligned} (1) & \text{ } R \text{ and } \Theta \text{ always finite,} \\ (2) & \text{ } R=0 \text{ and } \Theta=0 \text{ at infinity,} \\ \text{and } (3) & \text{ } R=W \cos \theta, \Theta=-W \sin \theta \text{ on the surface of the sphere,} \\ & \text{ i. e., when } \rho=a. \end{aligned} \right\} \quad (25)$$

The terms we have chosen are all such that R and Θ derived from them vanish at an infinite distance, and, if we take $C + D \cos \theta = b_0(1 - \cos \theta)$, are everywhere finite, since $\frac{\partial \psi}{\partial \theta}$ and $\frac{\partial \psi}{\partial \rho}$ are both divisible by $\sin \theta$. The last condition [25(3)] remains to be satisfied by a proper combination of the functions and determination of the constants.

Oseen's solution is equivalent to taking the terms in (23) whose coefficients are b_0 , L , and M ; that is to say,

$$\psi = b_0 e^{-k\rho(1+\cos\theta)} (1 - \cos \theta) + L \cos \theta + \frac{M \sin^2 \theta}{\rho}. \quad (26)$$

Using the conditions [25(3)] at the surface of the sphere to determine the constants, we have

$$\left. \begin{aligned} W \cos \theta &= -\frac{L}{a^2} + \frac{2M \cos \theta}{a^3} + \frac{b_0 e^{-ka(1+\cos\theta)}}{a^2} \{1 + ka(1 - \cos \theta)\}, \\ -W \sin \theta &= \frac{M \sin \theta}{a^3} + \frac{b_0 k}{a} e^{-ka(1+\cos\theta)} \sin \theta. \end{aligned} \right\} \quad (27)$$

Equating the constant terms in each of these two equations and the coefficients of $\cos \theta$ on each side of the first, we have

$$\frac{L}{a^2} = \frac{b_0}{a^2}, \quad W = \frac{2M}{a^3} - \frac{2b_0 k}{a}, \quad -W = \frac{M}{a^3} + \frac{b_0 k}{a}.$$

It is to be noticed that we have neglected all squares and higher powers of the quantity ka , which must therefore be assumed to be small. Hence we derive

$$L = b_0 = -\frac{3a\nu}{2}, \quad M = -\frac{Wa^3}{4}.$$

Substituting these values in (26), we have

$$\psi = -\frac{3a\nu}{2} [e^{-k\rho(1+\cos\theta)} (1 - \cos \theta) + \cos \theta] - \frac{Wa^3 \sin^2 \theta}{4\rho}. \quad (28)$$

If we transform this value of ψ into cylindrical coordinates, find u and w from equations (3), and resolve u in the directions of the x and y axes, we get expressions for the velocities which, except for notation, are exactly those derived by Oseen. As explained in section 4, this solution differs from Stokes' solution (7) fundamentally in that it satisfies the differential equation (9), which for large values of ρ is a close approximation to the complete equation (5). We can see its significance by throwing it into several different forms. In the first place, since a constant makes no difference in a function used solely in the

form of its derivatives, we may write it

$$\psi = \frac{Wa^2 \sin^2 \theta}{4} \left[3 \frac{e^{-k\rho(1+\cos \theta)} - 1}{-ka(1+\cos \theta)} - \frac{a}{\rho} \right]. \quad (28A)$$

Expanding the exponential term, this is

$$\psi = \frac{Wa^2}{4} \left(\frac{3\rho}{a} - \frac{a}{\rho} \right) \sin^2 \theta - \frac{3kaW}{8} \rho^2 \sin^2 \theta (1 + \cos \theta) + \dots \quad (28B)$$

The first term of this expansion is Stokes' solution; the following terms contain powers of $\frac{Wd\rho}{\mu}$ and are therefore less important than the first when ρ is small (*i. e.*, near the sphere) and W is small (*i. e.*, for slow motion). As Oseen says, then, in the neighborhood of the sphere the solution approaches that of Stokes as a limit when W approaches zero. It is important to notice, however, that the character of the solution is essentially different from that of Stokes. If we plot the stream-lines for Stokes' solution, we notice that we have symmetry with regard to the plane $z'=0$; for Oseen's, however, the stream-lines are more crowded following the sphere. If, for instance, we set the first two terms of (28) equal to a constant; that is, $e^{-k\rho(1+\cos \theta)}(1-\cos \theta) + \cos \theta = c$, the curves thus resulting, more conveniently expressed in the form

$$\rho = \frac{\log \frac{1-\cos \theta}{c-\cos \theta}}{k(1+\cos \theta)},$$

are asymptotic to the lines $\theta = \cos^{-1} c$ before the sphere and run off in a parabolic form in the wake of the sphere. The simplest way of seeing the difference between the conditions before and after the sphere, is to notice that the velocities will each have an exponential factor $-k\rho(1+\cos \theta)$; when θ is in the first quadrant, this term will be smaller than when θ is in the second quadrant, for the same values of ρ . Since this corresponds with our observations of fluid motion, the solution derived by Oseen appears worthy of further consideration and improvement.

Expanding the boundary conditions as given in (27), we have

$$\left. \begin{aligned} W \cos \theta &= -\frac{L}{a^2} + \frac{2M \cos \theta}{a^3} + \frac{b_0}{a^2} \left\{ 1 - ka(1+\cos \theta) + \frac{k^2 a^2}{2} (1+\cos \theta)^2 - \dots \right\} \\ &\quad \times \{ 1 + ka(1-\cos \theta) \}, \\ -W \sin \theta &= \frac{M \sin \theta}{a^3} + \frac{b_0 k \sin \theta}{a} \left\{ 1 - ka(1+\cos \theta) + \frac{k^2 a^2}{2} (1+\cos \theta)^2 - \dots \right\}. \end{aligned} \right\} \quad (27A)$$

From these equations it is clear that on the surface of the sphere the velocity components found in (28) differ from the velocity components of the sphere

by terms whose ratio to the terms $W \cos \theta$ and $-W \sin \theta$, is ka . We proceed to improve this approximation by using two more terms from (20), and by this means making the above-mentioned ratio equal to $k^2 a^2$.

9. Let

$$\psi = \sin^2 \theta \left\{ b_0 \frac{e^{-k\rho(1+\cos \theta)} - 1}{1 + \cos \theta} + b_1 \left(k + \frac{1}{\rho} \right) e^{-k\rho(1+\cos \theta)} + \frac{M}{\rho} + \frac{N \cos \theta}{\rho^2} \right\} \quad (29)$$

$$\begin{aligned} = \sin^2 \theta & \left\{ -b_0 k \rho + b_1 \left(k + \frac{1}{\rho} \right) + \frac{M}{\rho} - \frac{N}{\rho^2} \left[\frac{b_0 k^2 \rho^2}{2} - b_1 k (1 + k \rho) + \frac{N}{\rho^2} \right] (1 + \cos \theta) \right. \\ & + \left[-\frac{b_0 k^3 \rho^3}{3} + \frac{b_1 k^2 \rho^2}{2} \left(k + \frac{1}{\rho} \right) \right] (1 + \cos \theta)^2 + \dots \\ & \left. + \left[b_0 \frac{(-k\rho)^{n+1}}{n+1} + \frac{b_1 (-k\rho)^n}{n} \left(k + \frac{1}{\rho} \right) \right] (1 + \cos \theta)^n + \dots \right\} \quad (29A) \end{aligned}$$

$$= \sin^2 \theta [B_0 + B_1 (1 + \cos \theta) + B_2 (1 + \cos \theta)^2 + \dots + B_n (1 + \cos \theta)^n + \dots]. \quad (29B)$$

The boundary conditions (21) and (22) at the surface of the sphere give

$$R = \frac{1}{\rho^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = W \cos \theta,$$

$$\Theta = -\frac{1}{\rho \sin \theta} \frac{\partial \psi}{\partial \rho} = -W \sin \theta, \text{ when } \rho = a.$$

As the first of these is to be true, for all values of θ , we may integrate, giving

$$\psi = \frac{1}{2} W a^2 \sin^2 \theta, \text{ when } \rho = a. \text{ We have, therefore,}$$

$$\frac{1}{2} W a^2 \sin^2 \theta = \sin^2 \theta [B_0 + B_1 (1 + \cos \theta) + \dots],$$

$$W a \sin^2 \theta = \sin^2 \theta \left[\frac{\partial B_0}{\partial \rho} + \frac{\partial B_1}{\partial \rho} (1 + \cos \theta) + \dots \right].$$

Dividing by $\sin^2 \theta$ and equating like powers of $(1 + \cos \theta)$, we obtain

$$\left. \begin{aligned} \frac{1}{2} W a^2 &= B_0 \Big|_{\rho=a}, & W a &= \frac{\partial B_0}{\partial \rho} \Big|_{\rho=a}, \\ 0 &= B_n \Big|_{\rho=a}, & 0 &= \frac{\partial B_n}{\partial \rho} \Big|_{\rho=a}. \end{aligned} \right\} \quad (30)$$

Using the values of B_0, B_1 , etc., given above in equation (29A), we find that we can satisfy the four equations for $B_0, B_1, \frac{\partial B_0}{\partial \rho}$ and $\frac{\partial B_1}{\partial \rho}$, neglecting squares and higher powers of ka , with the following values for the constants:

$$\left. \begin{aligned} b_0 &= -\frac{3}{4} W a^2 \left(\frac{1}{ka} + \frac{3}{4} \right), & M &= \frac{W a^3}{2} \left(1 - \frac{3}{2} ka \right), \\ b_1 &= -\frac{3}{4} W a^3 \left(1 - \frac{3}{4} ka \right), & N &= -\frac{3}{8} W a^4 \cdot ka. \end{aligned} \right\} \quad (31)$$

To satisfy the boundary conditions (30) exactly, we should have B_n and $\frac{\partial B_n}{\partial \rho} = 0$ when $\rho = a$; it is obvious that, while the given values of the constants do not effect this, the terms which should but do not vanish all contain at least the square of ka as a factor, and our error in approximately satisfying the boundary conditions at the sphere is proportional to the square instead of the first power of the small quantity ka . We see, therefore, that while solution (28) holds as valid in the limit when ka approaches zero, the solution just found is a better approximation when ka is sufficiently small so that its square, but not its first power, may be neglected. If a closer approximation is desired, two more terms may be added, namely:

$$b_2 e^{-k\rho(1+\cos\theta)} \sin^2\theta \cos\theta \left(k^2 + \frac{3k}{\rho} + \frac{3}{\rho^2} \right) + \frac{P \sin^2\theta (5 \cos^2\theta - 1)}{\rho^3}, \quad (32)$$

and we shall then be able to satisfy exactly the boundary conditions $B_2 = 0$, $\frac{\partial B_2}{\partial \rho} = 0$ when $\rho = a$, making our error proportional to $k^3 a^3 W$. Similarly, we can add any number of pairs of functions and make our error proportional to as high a power of ka as we desire. On account also of the $\frac{1}{n}$ in the denominator of the n -th term of (29), this error is more and more negligible as we go on. We have shown, then, how to obtain as accurate a solution of equation (9) as may be desired.

10. In this differential equation (9) which we have just discussed, we have neglected certain terms which were included in our fundamental equation (5). We shall now include these and proceed to obtain a solution of the complete equation (5) accurate as far as the terms involving $k^2 a^2$ or $k^2 \rho^2$ as a factor. When transformed to polar coordinates, equation (5) becomes

$$\begin{aligned} & \left[\frac{1}{\rho^2 \sin\theta} \left(\frac{\partial\psi}{\partial\theta} \frac{\partial}{\partial\rho} - \frac{\partial\psi}{\partial\rho} \frac{\partial}{\partial\theta} \right) + \frac{2}{\rho^2 \sin^2\theta} \left(\cos\theta \frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho} \frac{\partial\psi}{\partial\theta} \right) \right. \\ & \quad \left. - \nu \left\{ \frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \right\} \right. \\ & \quad \left. - W \left(\cos\theta \frac{\partial}{\partial\rho} - \frac{\sin\theta}{\rho} \frac{\partial}{\partial\theta} \right) \right] \left[\frac{\partial^2\psi}{\partial\rho^2} + \frac{\sin\theta}{\rho^2} \frac{\partial}{\partial\theta} \left(\frac{\partial\psi}{\sin\theta \partial\theta} \right) \right] = 0. \quad (5A) \end{aligned}$$

Let us assume that the desired approximate solution of this equation is of the form

$$\psi = \sin^2\theta [A_0 + A_1 \cos\theta], \quad (33)$$

where A_0 and A_1 are functions of ρ , and A_1 contains ka as a factor. This assumption appears appropriate in view of the fact that, to the desired degree

of approximation, each of the solutions (7), (28) and (29) already found is of this type. Substituting this value of ψ in the differential equation (5A), omitting all terms which contain $k^2 a^2$ as a factor, we have, after some simplification,

$$\begin{aligned} & \left(\frac{2A_0}{\rho^2} - W \right) \left\{ \frac{\partial}{\partial \rho} \left(\frac{\partial^2 A_0}{\partial \rho^2} - \frac{2A_0}{\rho^2} \right) - \frac{2}{\rho} \left(\frac{\partial^2 A_0}{\partial \rho^2} - \frac{2A_0}{\rho^2} \right) \right\} \cos \theta \\ & - \nu \left[\frac{\partial^4 A_0}{\partial \rho^4} - \frac{4}{\rho^2} \frac{\partial^2 A_0}{\partial \rho^2} + \frac{8}{\rho^3} \frac{\partial A_0}{\partial \rho} - \frac{8A_0}{\rho^4} \right. \\ & \quad \left. + \left(\frac{\partial^4 A_1}{\partial \rho^4} - \frac{12}{\rho^2} \frac{\partial^2 A_1}{\partial \rho^2} + \frac{24}{\rho^3} \frac{\partial A_1}{\partial \rho} \right) \cos \theta \right] = 0. \end{aligned} \quad (34)$$

In order to satisfy this relation, we must in the first place have

$$\frac{\partial^4 A_0}{\partial \rho^4} - \frac{4}{\rho^2} \frac{\partial^2 A_0}{\partial \rho^2} + \frac{8}{\rho^3} \frac{\partial A_0}{\partial \rho} - \frac{8A_0}{\rho^4} = 0. \quad (35)$$

That is to say, any value assumed for A_0 must be such that $A_0 \sin^2 \theta$ satisfies $D^2 \psi = 0$, since (35) results from this part of (5A), which was used by Stokes. The complete solution of this equation can easily be shown to be

$$A_0 = \frac{a_0}{\rho} + a_1 \rho + a_2 \rho^2 + a_3 \rho^4.$$

In order to satisfy (34), we must also have

$$\frac{1}{\nu} \left(\frac{2A_0}{\rho^2} - W \right) \left(\frac{\partial}{\partial \rho} - \frac{2}{\rho^2} \right) \left(\frac{\partial^2 A_0}{\partial \rho^2} - \frac{2A_0}{\rho^2} \right) = \frac{\partial^4 A_1}{\partial \rho^4} - \frac{12}{\rho^2} \frac{\partial^2 A_1}{\partial \rho^2} + \frac{24}{\rho^3} \frac{\partial A_1}{\partial \rho}. \quad (36)$$

Hence we may easily determine the value of A_1 corresponding to any value of A_0 .

11. The first obvious assumption is to take A_0 from Stokes' solution (7),

$$A_0 = b_0 \left(\frac{3\rho}{a} - \frac{a}{\rho} \right).$$

Then

$$\frac{\partial^2 A_0}{\partial \rho^2} - \frac{2A_0}{\rho^2} = -\frac{6b_0}{a\rho}, \quad (37)$$

and (34) becomes

$$\rho^4 \frac{\partial^4 A_1}{\partial \rho^4} - 12\rho^2 \frac{\partial^2 A_1}{\partial \rho^2} + 24\rho \frac{\partial A_1}{\partial \rho} = \frac{18b_0}{\nu a} \left(\frac{6b_0\rho}{a} - \frac{2b_0 a}{\rho} - W\rho^2 \right). \quad (38)$$

Assume a solution,

$$A_1 = d_1 \rho^2 + d_2 \rho + d_3 + \frac{d_4}{\rho} + \frac{d_5}{\rho^2}, \quad (39)$$

in which the third and fifth terms are parts of the complementary function, the fifth term being essentially ψ_2 of equation (17). We could now easily

obtain the value of A_1 ; our work is simplified, however, by first determining b_0 . As our solution must satisfy the boundary conditions at the surface of the sphere, we have, as in (25),

$$\left. \frac{1}{a^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \right]_{\rho=a} = W \cos \theta, \quad \left. \frac{1}{a \sin \theta} \frac{\partial \psi}{\partial \rho} \right]_{\rho=a} = W \sin \theta. \quad (40)$$

Since $\psi = \sin^2 \theta (A_0 + A_1 \cos \theta)$,

$$W a^2 \cos \theta = 2 A_0 \cos \theta + A_1 [-1 + 3 \cos^2 \theta],$$

$$W a = \frac{\partial A_0}{\partial \rho} + \frac{\partial A_1}{\partial \rho} \cos \theta.$$

We must therefore have

$$\left. \begin{aligned} \frac{W a^2}{2} = A_0 \right]_{\rho=a}, \quad W a = \frac{\partial A_0}{\partial \rho} \right]_{\rho=a}, \\ A_1 \right]_{\rho=a} = 0, \quad \frac{\partial A_1}{\partial \rho} \right]_{\rho=a} = 0. \end{aligned} \right\} \quad (41)$$

From these conditions on A_0 , we find $b_0 = \frac{1}{4} W a^2$. Substituting this value for b_0 , (38) becomes

$$\rho^4 \frac{\partial^4 A_1}{\partial \rho^4} - 12 \rho^2 \frac{\partial^2 A_1}{\partial \rho^2} + 24 \rho \frac{\partial A_1}{\partial \rho} = \frac{9 W^2 a}{4 \nu \rho} [-2 \rho^3 + 3 a \rho^2 - a^3]. \quad (42)$$

Hence we easily obtain

$$A_1 = -\frac{3}{32} \frac{W^2 a}{\nu \rho^2} [2 \rho^4 - 3 a \rho^3 + d_3 \rho^2 a^2 - a^3 \rho + d_5]. \quad (43)$$

Using (41), we have

$$d_3 = d_5 = 1. \quad (44)$$

$$\therefore A_1 = -\frac{3}{32} \frac{W^2 a}{\nu} \left(1 - \frac{a}{\rho}\right)^2 (2 \rho^2 + a \rho + a^2).$$

The solution given by this assumption of the value of A_0 is therefore

$$\psi = \frac{1}{4} W a \sin^2 \theta \left\{ \left(3 \rho - \frac{a^2}{\rho}\right) - \frac{3 W}{8 \nu} \left(1 - \frac{a}{\rho}\right)^2 (2 \rho^2 + a \rho + a^2) \cos \theta \right\}, \quad (45)$$

which is exactly the correction found by Noether, as stated in (10); the velocities derived from this are given in (8), and were first found by Whitehead. The objection which Whitehead himself raised against this approximation still stands, however; that is, the velocities do not vanish at infinity as required by the conditions of the problem.

12. In order to avoid this difficulty at infinity, let us take A_0 from the solution of an equation which is, as Stokes' is not, a good approximation at a

large distance from the sphere. The simplest one of this type is (28). We shall consider this as our basis and determine a correction, accurate as far as terms involving $k^2 a^2$ or $k^2 \rho^2$ as a factor, which shall take account of the quadratic terms. Accordingly we take

$$A_0 = b_0 \left(\frac{3\rho}{a} - \frac{a}{\rho} \right) - \frac{3}{8} ka W \rho^2. \quad (46)$$

Determining b_0 at once from equations (41), we have

$$\begin{aligned} \frac{W a^2}{2} &= 2b_0 - \frac{3}{8} ka W a^2, \\ W a &= \frac{4b_0}{a} - \frac{3}{4} ka W a; \end{aligned}$$

that is

$$b_0 = \frac{W a^2}{4} \left(1 + \frac{3}{4} ka \right).$$

Substituting the value of A_0 from (46) in (36), and neglecting, as we are doing throughout, all terms involving the square of ka , we again obtain (42). The approximate solution $\psi = \sin^2 \theta (A_0 + A_1 \cos \theta)$ derived by this method is then

$$\begin{aligned} \psi = \frac{1}{4} W \sin^2 \theta \left[\left(1 + \frac{3}{4} ka \right) \left(3\rho a - \frac{a^3}{\rho} \right) - \frac{3ka}{2} \rho^2 \right. \\ \left. - \frac{3}{4} ka \cos \theta \left(1 - \frac{a}{\rho} \right)^2 (2\rho^2 + a\rho + a^2) \right], \quad (47) \end{aligned}$$

which is in substance exactly equation (11), as derived by Oseen. In this form, however, the solution is still subject to the objection that it gives velocities which do not vanish at infinity.

Solution (47) is therefore unsatisfactory; but if we notice that the correction taking into account the quadratic terms is based on solution (28), and must therefore be added to it to give the required result, this difficulty disappears. The term $-\frac{3}{8} Wka\rho^2 \sin \theta \cos \theta$ of (47) is already included in the exponential term of (28); combining with (28) the terms of (47) not already included, we obtain

$$\begin{aligned} \psi = -\frac{3\nu a}{2} \left(1 + \frac{3}{4} ka \right) \sin^2 \theta \frac{e^{-k\rho(1+\cos \theta)} - 1}{1 + \cos \theta} \\ - \frac{W a^3}{4} \left(1 + \frac{3}{4} ka \right) \frac{\sin^2 \theta}{\rho} + \frac{3Wka}{16} \sin^2 \theta \cos \theta \left(3a\rho - a^2 + \frac{a^3}{\rho} - \frac{a^4}{\rho^2} \right). \quad (48) \end{aligned}$$

Hence, as usual,

$$R = \frac{1}{\rho^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = -\frac{3\nu a}{2\rho^2} \left(1 + \frac{3}{4}ka\right) [e^{-k\rho(1+\cos \theta)} \{1 + k\rho(1 - \cos \theta)\} - 1] \\ - \frac{Wa^3}{2} \left(1 + \frac{3}{4}ka\right) \frac{\cos \theta}{\rho^3} + \frac{3Wka}{16\rho^2} (-1 + 3 \cos^2 \theta) \left(3a\rho - a^2 + \frac{a^3}{\rho} - \frac{a^4}{\rho^2}\right), \quad (49)$$

$$\Theta = -\frac{1}{\rho \sin \theta} \frac{\partial \psi}{\partial \rho} = -\frac{3Wa}{4} \left(1 + \frac{3}{4}ka\right) \sin \theta e^{-k\rho(1+\cos \theta)} \\ - \frac{Wa^3}{4} \left(1 + \frac{3}{4}ka\right) \frac{\sin \theta}{\rho^3} - \frac{3Wka}{16\rho} \sin \theta \cos \theta \left[3a - \frac{a^3}{\rho^2} + \frac{2a^4}{\rho^3}\right]. \quad (50)$$

Using these velocities, we find with Oseen that Stokes' law should be

$$F = 6\pi\mu Wa \left(1 + \frac{3Wda}{8\mu}\right).$$

As is obvious, these velocities, (49) and (50), both vanish when ρ is infinite; when $\rho = a$, they are, except for terms involving the squares or higher powers of ka , equal to $W \cos \theta$ and $-W \sin \theta$, respectively; and since, omitting squares and higher powers of ka and $k\rho$, (48) is the same as (47), to that approximation they satisfy the complete differential equation (5). The solution, then, is superior to any previously given. The difficulty first raised by Whitehead, that a certain term gives velocities which do not vanish at infinity, is met by showing that this term is included in an exponential expression.